# 3 Sobolev Spaces

**Exercise 3.1.** Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\infty}(I)$ be a Cauchy sequence. Then $\{u_n\}_{n\in\mathbb{N}}$ and $\{u_n'\}_{n\in\mathbb{N}}$ are Cauchy sequences in $L^\infty(I)$, thus $u_n \to u$ and $u_n' \to g$ in $L^\infty(I)$. Moreover, since $u_n \in W^{1,\infty}(I)$, we have

$$u_n(x) = u_n(y) + \int_y^x u_n'(t)\,dt, \quad \forall n \in \mathbb{N}.$$

By letting $n \to \infty$ we obtain that the limits $u$ and $g$ satisfy

$$u(x) = u(y) + \int_y^x g(t)\,dt$$

and from Exercise 3.1 of Sheet 8, we also get that $g = u'$ in $\mathcal{D}'(I)$. This shows that $u \in W^{1,\infty}(I)$ as desired.

**Exercise 3.2.** Since $u \in W^{1,p}(I)$ we have $u(x) = u(0) + \displaystyle\int_0^x u'(t)\,dt$. Note that

$$\frac{1}{x}\int_0^x |u'(t)|\,dt \leq \frac{1}{x}x^{1-\frac{1}{p}}\|u'\|_{L^p(I)} = \frac{\|u'\|_{L^p(I)}}{x^{\frac{1}{p}}} \in L^1(I),$$

thus $\dfrac{u(x)}{x} \in L^1(I)$ if and only if $\dfrac{u(0)}{x} \in L^1(I)$ and this holds true if and only if $u(0) = 0$.

Moreover let $v(x) = (1 + |\log(x)|)^{-1} = (1 - \log(x))^{-1}$. We have $v'(x) = x^{-1}(1 - \log(x))^{-2}$, from which we get

$$\int_0^1 v(x)\,dx = \int_{-\infty}^0 \frac{e^y}{1-y}\,dy < \infty$$
$$\int_0^1 v'(x)\,dx = \int_{-\infty}^0 \frac{1}{(1-y)^2}\,dy < \infty.$$

Moreover $v(0) = 0$ but

$$\int_0^1 \frac{v(x)}{x}\,dx = \int_{-\infty}^0 \frac{1}{1-y}\,dy = \infty.$$

**Exercise 3.3.** We define

$$u_n(x) = \begin{cases} 0, & \text{if } 0 < x \leq \frac{1}{2} \\ n(x - \frac{1}{2}), & \text{if } \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n} \\ 1, & \text{if } \frac{1}{2} + \frac{1}{n} < x < 1. \end{cases}$$

Obviously $u_n \in L^1(I)$ for every $n \in \mathbb{N}$, and moreover its distributional derivative $u_n'$ is given by

$$u_n'(x) = \begin{cases} 0, & \text{if } 0 < x \leq \frac{1}{2} \\ n, & \text{if } \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n} \\ 0, & \text{if } \frac{1}{2} + \frac{1}{n} < x < 1. \end{cases}$$

Thus, also $u_n' \in L^1(I)$. Moreover

$$\|u_n\|_{W^{1,1}(I)} = \|u_n\|_{L^1(I)} + \|u_n'\|_{L^1(I)} \leq \frac{1}{2} + 1 = \frac{3}{2},$$

but $u_n(x) \to H(x - \frac{1}{2})$ almost everywhere, and since $\{u_n\}_{n\in\mathbb{N}} \subset C^0(I)$, $\{u_n\}_{n\in\mathbb{N}}$ can not converge in $L^\infty(I)$, because its limit would be a continuous function and $H(x - \frac{1}{2})$ is clearly not.

**Exercise 3.4.** Suppose that $u \in \mathrm{Lip}(I)$. By Rademacher's Theorem $u$ is differentiable almost everywhere and its distributional derivative coincides with the almost everywhere one. But since $u \in \mathrm{Lip}(I)$, the almost everywhere derivative is bounded, indeed for almost every $x \in I$

$$|u'(x)| \leq \lim_{h\to 0} \frac{|u(x+h) - u(x)|}{|h|} \leq \lim_{h\to 0} \frac{L|h|}{|h|} = L.$$

Thus, also the distributional derivative $u' \in L^\infty(I)$ and this shows that $\mathrm{Lip}(I) \subset W^{1,\infty}(I)$.

Let now $u \in W^{1,\infty}(I)$. We have

$$|u(x) - u(y)| \leq \left| \int_y^x u'(t)\, dt \right| \leq \|u'\|_{L^\infty(I)} |x - y|,$$

which implies that $u \in \mathrm{Lip}(I)$.

**Exercise 3.5.** Let $\varphi \in C_0^\infty(\mathbb{R})$ and $u_n(x) = \varphi(x + n)$. Clearly $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R})$ is bounded, indeed

$$\|u_n\|_{\mathrm{W}^{1,p}(\mathbb{R})} = \|\varphi\|_{\mathrm{W}^{1,p}(\mathbb{R})}$$

Moreover, since $\varphi$ vanishes at infinity, $u_n(x) \to 0$ for every $x \in \mathbb{R}$, but $\|u_n\|_{L^q(\mathbb{R})} = \|\varphi\|_{L^q(\mathbb{R})} \neq 0$. Thus there cannot be a subsequence converging strongly in $L^q(\mathbb{R})$.

Let now $p' = \dfrac{p}{p-1}$. Since $C_c^\infty(\mathbb{R})$ is dense in $L^{p'}(\mathbb{R})$, for every $v \in L^{p'}(\mathbb{R})$ and for every $\varepsilon > 0$ there exists $v_\varepsilon \in C_c^\infty(\mathbb{R})$ such that

$$\|v_\varepsilon - v\|_{\mathrm{L}^{p'}(\mathbb{R})} < \frac{\varepsilon}{2\,\|\varphi\|_{\mathrm{L}^p(\mathbb{R})}} \tag{1}$$

Moreover, for every $\varepsilon > 0$, $|u_n(x)v_\varepsilon(x)| \leq \|\varphi\|_{L^\infty(\mathbb{R})}|v_\varepsilon(x)| \in L^1(\mathbb{R})$, and since $\varphi$ vanishes at infinity, we also get $u_n(x)v_\varepsilon(x) \to 0$ for every $x \in \mathbb{R}$, as $n \to \infty$. Thus, by Lebesgue dominated convergence, for every $\varepsilon > 0$, there exists $N_\varepsilon$ such that $\forall n \geq N_\varepsilon$

$$\left|\int_{\mathbb{R}} u_n(x)v_\varepsilon(x)\,dx\right| < \frac{\varepsilon}{2}. \tag{2}$$

Finally, by using (1) and (2), we can conclude

$$\begin{aligned}\left|\int_{\mathbb{R}} u_n(x)v(x)\,dx\right| &\leq \int_{\mathbb{R}} |u_n(x)||v(x)-v_\varepsilon(x)|\,dx + \left|\int_{\mathbb{R}} u_n(x)v_\varepsilon(x)\,dx\right| \\ &\leq \|u_n\|_{\mathrm{L}^p(\mathbb{R})}\|v-v_\varepsilon\|_{\mathrm{L}^p(\mathbb{R})} + \frac{\varepsilon}{2} < \|\varphi\|_{\mathrm{L}^p(\mathbb{R})}\frac{\varepsilon}{2\|\varphi\|_{\mathrm{L}^p(\mathbb{R})}} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

which shows that $u_n \rightharpoonup 0$ in $L^p(\mathbb{R})$. From the very same proof we also get $u_n' \rightharpoonup 0$ in $L^p(\mathbb{R})$.